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LETTER TO THE EDITOR

Bulk and boundary g_2 factorized S-matrices

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Abstract
We investigate the g_2-invariant bulk (1+1D, factorized) S-matrix constructed by Ogievetsky, using the bootstrap on the three-point coupling of the vector multiplet to constrain its CDD ambiguity. We then construct the corresponding boundary S-matrix, demonstrating it to be consistent with $Y(g_2, a_1 \times a_1)$ symmetry.

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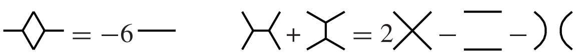
1. Introduction

As a preliminary step in the investigation using tensor methods of (1+1)-dimensional factorized S-matrices with exceptional $\mathbf{g}$ (and Yangian $Y(\mathbf{g})$) invariance, we investigate the case $\mathbf{g} = g_2$. The factorized S-matrix for the seven-dimensional representation **7** of g_2 was constructed by Ogievetsky [1], and we use this to construct the $(g_2 \times g_2)$-invariant S-matrix, applicable in the principal chiral model (PCM). We may choose the S-matrix to have a bootstrap pole for the self-coupling of the **7** multiplet, and the bootstrap applied to this process constrains the CDD factor.

We then investigate the corresponding solutions of the boundary Yang–Baxter equations and the boundary S-matrices for the g_2 PCM—that is, the extension to g_2 of the calculations carried out for classical $\mathbf{g}$ in [2]. The spectral decomposition is precisely that expected from the $Y(g_2, a_1 \times a_1)$ symmetry [3].

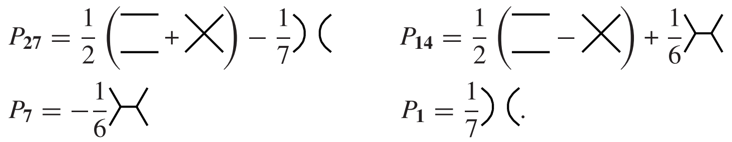
The method used is the diagrammatic technique of Cvitanovic [4]. We denote the cubic antisymmetric invariant of g_2 as , then construct the **7** of g_2 by taking the defining representation of $SO(7)$ and restricting to those $\in SO(7)$ such that  = . Here  satisfies the identities

 (1.1)

and it is (sometimes repeated) application of these which is needed to carry out the calculations. If we take the tensor product

$$\mathbf{7} \otimes \mathbf{7} = \mathbf{27} \oplus \mathbf{14} \oplus \mathbf{7} \oplus \mathbf{1}$$

then the projectors onto the irreducible g_2-representations are



2. The bulk S-matrix

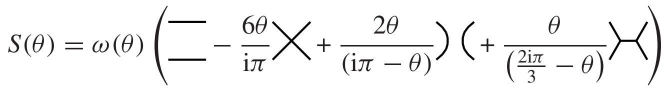
The following g_2-invariant S-matrix satisfies the Yang–Baxter equation [1, 5]:

$$S_{(1,1)}(\theta) = S(\theta) = \sigma(\theta)\,(P_{\mathbf{27}} + [2]P_{\mathbf{14}} + [8]P_{\mathbf{7}} + [2][12]P_{\mathbf{1}})$$

where $\sigma(\theta)$ is a scalar prefactor and

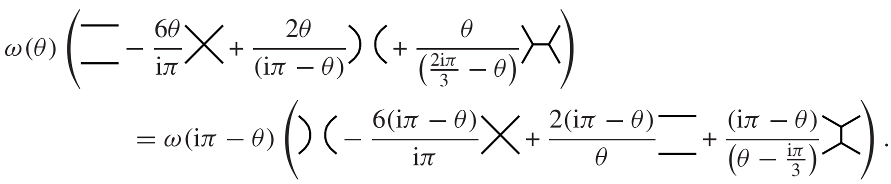
$$[y] = \frac{\frac{y\mathrm{i}\pi}{12} + \theta}{\frac{y\mathrm{i}\pi}{12} - \theta}.$$

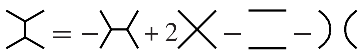
Imposing R-matrix unitarity on the S-matrix gives $\sigma(\theta)\sigma(-\theta) = 1$, and imposing Hermitian analyticity gives $\sigma(\theta) = \sigma(-\theta^*)^*$. We rewrite the S-matrix as



(2.1)

where $\sigma(\theta) = \left(1 - \frac{6\theta}{\mathrm{i}\pi}\right)\omega(\theta)$, and impose crossing symmetry



Using  we find that this is satisfied if

$$\omega(\theta) = \omega(\mathrm{i}\pi - \theta)\frac{(\mathrm{i}\pi - \theta)\left(\frac{2\mathrm{i}\pi}{3} - \theta\right)}{\theta\left(\frac{\mathrm{i}\pi}{3} - \theta\right)} \quad \Leftrightarrow \quad \sigma(\theta) = \sigma(\mathrm{i}\pi - \theta)\frac{(\mathrm{i}\pi - \theta)\left(\frac{2\mathrm{i}\pi}{3} - \theta\right)\left(\frac{\mathrm{i}\pi}{6} - \theta\right)}{\theta\left(\frac{\mathrm{i}\pi}{3} - \theta\right)\left(\theta - \frac{5\mathrm{i}\pi}{6}\right)}.$$

To solve for $\sigma(\theta)$ we first introduce

$$\mu_a(\theta) = \frac{\Gamma\left(\frac{\theta}{2\mathrm{i}\pi} + \frac{a}{12}\right)\Gamma\left(\frac{-\theta}{2\mathrm{i}\pi} + \frac{a}{12} + \frac{1}{2}\right)}{\Gamma\left(\frac{-\theta}{2\mathrm{i}\pi} + \frac{a}{12}\right)\Gamma\left(\frac{\theta}{2\mathrm{i}\pi} + \frac{a}{12} + \frac{1}{2}\right)}$$

which satisfies $\mu_a(\theta)\mu_a(-\theta) = 1$ and, for real a, $\mu_a(\theta) = \mu_a(-\theta^*)^*$. Further

$$\frac{\mu_a(\theta)}{\mu_a(\mathrm{i}\pi - \theta)} = \frac{\left(\frac{a\mathrm{i}\pi}{6} - \theta\right)}{\left(\theta - \mathrm{i}\pi + \frac{a\mathrm{i}\pi}{6}\right)} = \frac{\mu_{6-a}(\mathrm{i}\pi - \theta)}{\mu_{6-a}(\theta)}.$$

We seek a minimal S-matrix, with no poles on the physical strip. The factor $\mu_a(\theta)$ has simple poles at $\theta = -2\mathrm{i}\pi n - \frac{a\mathrm{i}\pi}{6}$, $\theta = 2\mathrm{i}\pi n + \mathrm{i}\pi + \frac{a\mathrm{i}\pi}{6}$ and simple zeroes at $\theta = 2\mathrm{i}\pi n + \frac{a\mathrm{i}\pi}{6}$, $\theta = -2\mathrm{i}\pi n - \mathrm{i}\pi - \frac{a\mathrm{i}\pi}{6}$ for $n = 0, 1, 2, \ldots$. Thus, to cancel the poles in (2.1) we are led to

$$\sigma(\theta) = \mu_0(-\theta)\mu_1(\theta)\mu_3(\theta)\mu_4(\theta)$$

so that

$$\sigma(\theta)=\frac{\Gamma\left(\frac{\theta}{2\mathrm{i}\pi}+\frac{1}{2}\right)\Gamma\left(\frac{-\theta}{2\mathrm{i}\pi}\right)\Gamma\left(\frac{-\theta}{2\mathrm{i}\pi}+\frac{7}{12}\right)\Gamma\left(\frac{\theta}{2\mathrm{i}\pi}+\frac{1}{12}\right)\Gamma\left(\frac{-\theta}{2\mathrm{i}\pi}+\frac{3}{4}\right)\Gamma\left(\frac{\theta}{2\mathrm{i}\pi}+\frac{1}{4}\right)\Gamma\left(\frac{-\theta}{2\mathrm{i}\pi}+\frac{5}{6}\right)\Gamma\left(\frac{\theta}{2\mathrm{i}\pi}+\frac{1}{3}\right)}{\Gamma\left(\frac{-\theta}{2\mathrm{i}\pi}+\frac{1}{2}\right)\Gamma\left(\frac{\theta}{2\mathrm{i}\pi}\right)\Gamma\left(\frac{\theta}{2\mathrm{i}\pi}+\frac{7}{12}\right)\Gamma\left(\frac{-\theta}{2\mathrm{i}\pi}+\frac{1}{12}\right)\Gamma\left(\frac{\theta}{2\mathrm{i}\pi}+\frac{3}{4}\right)\Gamma\left(\frac{-\theta}{2\mathrm{i}\pi}+\frac{1}{4}\right)\Gamma\left(\frac{\theta}{2\mathrm{i}\pi}+\frac{5}{6}\right)\Gamma\left(\frac{-\theta}{2\mathrm{i}\pi}+\frac{1}{3}\right)}$$

(in fact we may choose plus or minus this—our choice of the positive sign will not affect the S-matrix). Thus we have established a minimal S-matrix which is g_2 invariant.

The g_2 PCM S-matrix acts on multiplets which are representations of $(g_2\times g_2)$, and is constructed from two minimal S-matrices together with a CDD factor $X(\theta)$:

$$S^{PCM}_{(1,1)}(\theta)=X_{(1,1)}(\theta)(S(\theta)_L\otimes S(\theta)_R).$$

In order that $S^{PCM}_{(1,1)}(\theta)$ satisfy R-matrix unitarity and crossing-symmetry we require

$$X_{(1,1)}(\theta)X_{(1,1)}(-\theta)=1 \qquad \text{and} \qquad \frac{X_{(1,1)}(\theta)}{X_{(1,1)}(\mathrm{i}\pi-\theta)}=1.$$

To construct X we use

$$(y)=(y)_\theta=\frac{\sinh\left(\frac{\theta}{2}+\frac{y\mathrm{i}\pi}{24}\right)}{\sinh\left(\frac{\theta}{2}-\frac{y\mathrm{i}\pi}{24}\right)}$$

this satisfies

$$(y)_\theta(y)_{-\theta}=1 \qquad \frac{(y)_\theta}{(y)_{\mathrm{i}\pi-\theta}}=(2y)_{2\theta} \qquad \text{and} \qquad (y)=(y+24).$$

The natural choice might be $X=-(2)(4)(8)(10)$, where we have allowed two **7**s to fuse (via simple poles with positive residues) to form either a **7** (at $\theta=2\mathrm{i}\pi/3$) or a $\mathbf{14}\oplus\mathbf{1}$ (at $\theta=\mathrm{i}\pi/6$, yielding a multiplet of mass $2\cos(\pi/12)=\frac{1}{2}(\sqrt{6}+\sqrt{2})$ times the mass of the **7**). We must then check that the bootstrap equations are satisfied for the scattering of a **7** off a fused $\mathbf{7}\subset\mathbf{7}\otimes\mathbf{7}$ (an intricate calculation requiring much repeated application of (1.1)). The minimal S-matrix is consistent with this, but the CDD factor requires an extra factor $(6)^2$, and we must have

$$X_{(1,1)}(\theta)=-(2)(4)(6)^2(8)(10).$$

The apparent double pole at $\mathrm{i}\pi/2$ thus introduced is spurious: it is cancelled by a simple zero in each minimal S.

3. The boundary *S*-matrix

We now consider the half-line case. Following [2], we try a minimal boundary S-matrix of the form

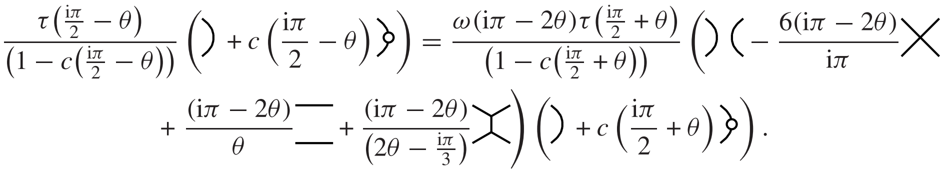
$$K(\theta)=\frac{\tau(\theta)}{(1-c\theta)}(\text{---}+c\theta\,{-\!\!\circ\!\!-}).$$

The conditions of boundary R-matrix unitarity and Hermitian analyticity impose the constraints

$$({-\!\!\circ\!\!-})^\dagger={-\!\!\circ} \qquad {-\!\!\circ\!\!-\!\!\circ\!\!-}=\text{---} \qquad c\in\mathrm{i}\mathbb{R} \qquad \tau(\theta)=\tau(-\theta^*)^* \qquad \text{and}$$

$$\tau(\theta)\tau(-\theta)=1.$$

We must also impose crossing-unitarity



After applying (1.1) we find that this implies

$$\frac{\tau\left(\frac{i\pi}{2}-\theta\right)}{\tau\left(\frac{i\pi}{2}+\theta\right)}=\frac{\omega(i\pi-2\theta)\left(1-c\left(\frac{i\pi}{2}-\theta\right)\right)\left(\theta-\frac{i\pi}{3}\right)}{\left(1-c\left(\frac{i\pi}{2}+\theta\right)\right)\left(2\theta-\frac{i\pi}{3}\right)}\left(14+2i\pi c\bigcirc+\frac{i\pi}{\theta}+4\left(c\bigcirc+\frac{6}{i\pi}\right)\theta\right)$$

$$\times\frac{\tau\left(\frac{i\pi}{2}-\theta\right)}{\tau\left(\frac{i\pi}{2}+\theta\right)}=\frac{\omega(i\pi-2\theta)\left(1-c\left(\frac{i\pi}{2}-\theta\right)\right)\left(\theta-\frac{i\pi}{3}\right)(i\pi+2\theta)}{\left(1-c\left(\frac{i\pi}{2}+\theta\right)\right)\left(2\theta-\frac{i\pi}{3}\right)}$$

$$\times\left(\frac{-\alpha}{\left(\theta-\frac{i\pi}{3}\right)}+\frac{1}{\theta}\mp\frac{12}{i\pi}\right)$$

together with (for non-trivial $-\!\circ\!-$) $\times\!\!\diamond = \pm\rangle$ and $\succ\!\!\diamond = \alpha\rangle$ for some constant α. Comparing the two expressions we find $\alpha=0$ and

$$c\bigcirc=-\frac{6(1\pm1)}{i\pi}.$$

However, $\bigcirc=0$ together with $-\!\circ\!-\!\circ\!-\;=\;—$ has no solutions in odd dimensions (the eigenvalues of such a matrix are ±1, an odd number of which cannot sum to zero). We thus have $(-\!\circ\!-)^t=-\!\circ\!-$ and

$$\frac{\tau\left(\frac{i\pi}{2}-\theta\right)}{\tau\left(\frac{i\pi}{2}+\theta\right)}=\frac{\omega(i\pi-2\theta)\left(1-c\left(\frac{i\pi}{2}-\theta\right)\right)\left(\theta-\frac{i\pi}{3}\right)(i\pi+2\theta)}{\left(1-c\left(\frac{i\pi}{2}+\theta\right)\right)\left(2\theta-\frac{i\pi}{3}\right)}\left(\frac{1}{\theta}-\frac{12}{i\pi}\right)$$

or

$$\frac{\tau\left(\frac{i\pi}{2}-\theta\right)}{\tau\left(\frac{i\pi}{2}+\theta\right)}=[6]\left[\frac{12}{ci\pi}-6\right]\sigma(2\theta)\qquad(-\!\circ\!-)^t=-\!\circ\!-\qquad\text{and}\qquad c\bigcirc=-\frac{12}{i\pi}.$$

Last we have to impose the boundary Yang–Baxter equation (bYBe). After some algebra we find that this is satisfied if

$$\succ+\;{}^{-\circ}_{-\circ}\!\!\succ=\frac{ci\pi}{12}\succ\!\!\langle\!\overset{\circ}{\ }\!\rangle\!-.$$

Now using (1.1) we find

$$-\langle\overset{\circ}{\ }\rangle-\;=\;-\!\circ\!-+\frac{12}{ci\pi}—.$$

Thus, putting these two results together

$${}^{-\circ}_{-\circ}\!\!\succ=\frac{ci\pi}{12}\succ\!-\!\circ\qquad\Leftrightarrow\qquad\succ=\frac{ci\pi}{12}\,\mathrm{\}\!\circ}.$$

Consequently we must have $c=\pm\frac{12}{i\pi}$, with $\}\!\circ=\pm\succ$ and $\bigcirc=\mp1$.

In summary, we have shown that the conditions of R-matrix unitarity, Hermitian analyticity, crossing unitarity and the bYBe are satisfied by a minimal boundary 'K'-matrix

$$\frac{\tau(\theta)}{\left(1\mp\frac{12\theta}{i\pi}\right)}\left(—\pm\frac{12\theta}{i\pi}-\!\circ\!-\right)=\tau(\theta)(P_-[\pm1]P_+)\qquad\left(P_\pm=\frac{1}{2}(—\pm-\!\circ\!-)\right)$$

where

$$(-\!\circ\!-)^\dagger=-\!\circ\!-\qquad(-\!\circ\!-)^t=-\!\circ\!-\qquad-\!\circ\!-\!\circ\!-=—\qquad\}\!\circ=\pm\succ\qquad\bigcirc=\mp1$$

$$\tau(\theta)\tau(-\theta)=1\qquad\tau(\theta)=\tau(-\theta^*)^*\qquad\frac{\tau\left(\frac{i\pi}{2}-\theta\right)}{\tau\left(\frac{i\pi}{2}+\theta\right)}=[6]\,[\pm1-6]\,\sigma(2\theta).$$

In fact, since $[1]_{\frac{i\pi}{2}-\theta}/[1]_{\frac{i\pi}{2}+\theta} = [-7]/[-5]$, the choice of sign is redundant—both choices give the same minimal K-matrix. We can write it as

$$\frac{\tau(\theta)}{\left(1-\frac{12\theta}{i\pi}\right)}\left(I+\frac{12\theta}{i\pi}E\right) = \tau(\theta)(P_-[1]P_+) \qquad \left(P_\pm = \frac{1}{2}(I \pm E)\right)$$

where $E = QXQ^{-1}$, $Q \in G_2$, $X = \mathrm{diag}(1, 1, 1, -1, -1, -1, -1)$. This is clearly a subspace of the symmetric space $SO(7)/S(O(3) \times O(4))$; in fact we have

$$E \in \frac{G_2}{SU(2) \times SU(2)}$$

the space of quaternionic subalgebras of the octonions, as may be seen by considering the action of G_2 on a basic triple of octonions [6].

The following constraints are imposed on $\tau(\theta)$:

$$\tau(\theta)\tau(-\theta) = 1 \qquad \tau(\theta) = \tau(-\theta^*)^* \qquad \frac{\tau\left(\frac{i\pi}{2}-\theta\right)}{\tau\left(\frac{i\pi}{2}+\theta\right)} = [6][-5]\sigma(2\theta).$$

To solve these we note that

$$\frac{\mu_a\left(\frac{i\pi}{2}-\theta\right)}{\mu_a\left(\frac{i\pi}{2}+\theta\right)} = -[2a-6]$$

and we define

$$\eta_a(\theta) = \frac{\Gamma\left(\frac{-\theta}{2i\pi}+\frac{a}{12}\right)\Gamma\left(\frac{\theta}{2i\pi}+\frac{a}{12}+\frac{1}{4}\right)}{\Gamma\left(\frac{\theta}{2i\pi}+\frac{a}{12}\right)\Gamma\left(\frac{-\theta}{2i\pi}+\frac{a}{12}+\frac{1}{4}\right)} \qquad \text{so that} \qquad \frac{\eta_a\left(\frac{i\pi}{2}-\theta\right)}{\eta_a\left(\frac{i\pi}{2}+\theta\right)} = \mu_{2a-6}(2\theta).$$

This leads us to

$$\tau(\theta) = \mu_{1/2}(\theta)\mu_6(\theta)\eta_{7/2}(\theta)\eta_{9/2}(\theta)\eta_5(\theta)\eta_6(\theta).$$

The simple poles of $\eta_a(\theta)$ are at $\theta = 2i\pi n + \frac{ai\pi}{6}$ and $\theta = -2i\pi n - \frac{i\pi}{2} - \frac{ai\pi}{6}$, while the simple zeroes are at $\theta = -2i\pi n - \frac{ai\pi}{6}$ and $\theta = 2i\pi n + \frac{i\pi}{2} + \frac{ai\pi}{6}$, and so the K-matrix is minimal.

The final piece we require for the complete PCM K-matrix is the factor $Y_1(\theta)$, which must satisfy

$$\frac{Y_1\left(\frac{i\pi}{2}-\theta\right)}{Y_1\left(\frac{i\pi}{2}+\theta\right)} = X_{(1,1)}(i\pi - 2\theta) = X_{(1,1)}(2\theta).$$

We make use of the fact that

$$\frac{(y)_{\frac{i\pi}{2}-\theta}}{(y)_{\frac{i\pi}{2}+\theta}} = (2y)_{i\pi-2\theta} = (2y+24)_{i\pi-2\theta}.$$

Thus the most natural choice is

$$Y_1(\theta) = (1)(2)(-9)^2(-8)(-7)(-6).$$

This has a physical strip simple pole at $\theta = \frac{i\pi}{12}$ at which the minimal K-matrix projects onto the subspace associated with P_+ (the smaller one, and the $(\mathbf{3}, \mathbf{1})$ of $(a_1 \times a_1)$ as found in [3]). The simple pole at $\theta = \frac{i\pi}{6}$ corresponds to an on-shell diagram which is possible precisely when the bulk 3-point coupling of **7**s exists.

We should also check the simpler trial solution of [2] for a minimal K-matrix, namely

$$K(\theta) = \rho(\theta)\,\text{—o—}.$$

Imposing crossing-unitarity gives

$$\frac{\rho\left(\frac{i\pi}{2}-\theta\right)}{\rho\left(\frac{i\pi}{2}+\theta\right)} = \omega(i\pi-2\theta)\left(\frac{4\left(\theta-\frac{i\pi}{3}\right)}{\left(2\theta-\frac{i\pi}{3}\right)} + \frac{4(i\pi-2\theta)(i\pi-3\theta)}{i\pi\left(2\theta-\frac{i\pi}{3}\right)} + \frac{(i\pi-2\theta)\left(\theta-\frac{i\pi}{3}\right)}{\theta\left(2\theta-\frac{i\pi}{3}\right)} + \frac{(i\pi-2\theta)}{\left(2\theta-\frac{i\pi}{3}\right)}\right)$$

which implies

$$\frac{\rho\left(\frac{i\pi}{2}-\theta\right)}{\rho\left(\frac{i\pi}{2}+\theta\right)} = \omega(i\pi-2\theta)\left(\frac{4\left(\theta-\frac{i\pi}{3}\right)}{\left(2\theta-\frac{i\pi}{3}\right)} + \frac{4(i\pi-2\theta)(i\pi-3\theta)}{i\pi\left(2\theta-\frac{i\pi}{3}\right)} + \frac{(i\pi-2\theta)\left(\theta-\frac{i\pi}{3}\right)}{\theta\left(2\theta-\frac{i\pi}{3}\right)} + \frac{(i\pi-2\theta)}{\left(2\theta-\frac{i\pi}{3}\right)}\right).$$

For non-trivial we must have $= 0$, $(\;)^t = \pm$ and $= \alpha$. But, as pointed out earlier, the constraint $= 0$ is inconsistent with $=$. Thus there are no non-trivial solutions of this form.

Acknowledgments

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